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On the trapping of waves along a discontinuity of depth in a rotating ocean

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It is shown that, according to the linearized theory of long waves in a rotating, unbounded sea, if there is a discontinuity in depth along a straight line separating two regions each of uniform depth, then wave motions may exist which are propagated along the discontinuity and whose amplitude falls off exponentially to either side. Thus the discontinuity acts as a kind of wave-guide.

The period of the waves is always greater than the inertial period. The wave period also exceeds the period of Kelvin waves in the deeper medium. As the ratio of the depth tends to infinity, the wave period tends to the inertial period or to the Kelvin wave period, whichever is the greater. On the other hand as the wavelength decreases (within the limits of shallow-water theory) so the waves tend to the non-divergent planetary waves found recently by Rhines.

In an infinite ocean of uniform depth free waves with period greater than a pendulum-day cannot normally be propagated without attenuation (if the Coriolis parameter is constant). But non-uniformities of depth provide a means whereby such energy may be channelled over great distances with little attenuation.

It is suggested that a gradually diminishing discontinuity will act as a chromatograph, each position along the discontinuity being marked by waves of a particular period.

1. Introduction

It is already well known that surface waves in water of finite depth may be trapped along a sloping beach (Stokes 1846; Ursell 1952; Eckart 1951) or may be totally reflected at a discontinuity in depth, with only an exponential fringe remaining on the deeper side (see, for example, Snodgrass, Munk & Miller 1962). These effects can be regarded as the result of wave refraction (Eckart 1951; for a recent discussion see Longuet-Higgins 1967) and take place even in the absence of Coriolis forces due to rotation.

On the other hand it can be shown that the effect of rotation on waves of a given frequency is to produce a decrease in the square of the wave-number to such an extent that if the Coriolis parameter exceeds the wave frequency the wave-number can no longer be real; the wave amplitude must diminish exponentially in at least one horizontal direction.

These two circumstances lead one to speculate as to the possibility of waves being trapped along a discontinuity, with an exponential decay on both sides, the decay on the deeper side being due essentially to wave refraction and that on the shallower side being due to the rotation.

However, the motion on the deeper side will also be affected by the rotation, and it then becomes a delicate question whether all the necessary conditions, including the appropriate boundary conditions at the discontinuity itself, can be met.

Now the existence of a sloping bottom, particularly a discontinuity in depth, will tend to restrict the horizontal component of the particle motion on the *deeper* side. Moreover, we know from the example of Kelvin waves (Thomson 1879) that the effect of restricting the horizontal component of displacement normal to the direction of wave propagation is to produce an attenuation of the wave amplitude to the *left* of the direction of wave propagation (in the northern hemisphere); for then the slope of the wave surface along the crests is such as to balance the Coriolis forces arising from the forwards particle motion at the crests; and similarly for the wave troughs. Hence the effect of rotation on the waves on the deeper side of the discontinuity will be to increase the trapping effect when the boundary is on the *right* of the direction of wave propagation, and to decrease it if the boundary is on the left. The former situation is clearly more likely to result in the type of motion in which we are interested.

As for the fluid in the shallower depth of water, the discontinuity in depth will have no such restraining effect on the normal component of the velocity. Rather, the deeper water will act as a release. Hence, it may be possible for the wave amplitude on the shallower side to decay in the opposite sense, as desired.

At this stage then we can foresee that doubly trapped wave motions are most likely to occur under the following conditions: (i) the wave period exceeds the period of inertial waves (one pendulum-day); (ii) the direction of propagation of the waves is the same as that of Kelvin waves in the deeper water, with the discontinuity to the right of the direction of wave propagation.

Further, if the water on the shallower side of the barrier were replaced by a rigid block, this would impose a restraint on the fluid motion on the deeper side. Hence by a general physical principle we expect that (iii) the wave period will exceed the period of Kelvin waves of the same wavelength propagated in the deeper water.

In the following paper these predictions are investigated analytically on the basis of the theory of long waves of infinitesimal amplitude in a rotating sea. All three of the above statements are verified, that is to say it is found that the trapped waves do exist analytically and that they always fulfill the three assertions made.

In one limiting case, that of zero depth on the shallower side, the waves reduce to Kelvin waves on the deeper side. In a second limiting case, that of zero divergence (or short wavelength, within the limits of the theory) the waves reduce to non-divergent planetary waves of the type discovered recently by Rhines (see Bretherton, Carrier & Longuet-Higgins 1966, p. 404).

A theory of low-frequency waves trapped along a continental shelf of *finite* width was recently developed by Robinson (1964). Possible generating mechanisms for such waves have been suggested by Mysak (1967) and Buchwald (unpublished). It now appears that for the existence of such waves the finite width of the shelf was not always essential.

The analysis is given in §§ 2 and 3 of the present paper and in the appendix. In § 4 we examine briefly the related problem of the reflexion and transmission of waves at a discontinuity. The conclusions, including possible implications for propagation of wave energy in the oceans, are summarized in § 5.

2. Basic equations

Let x and y be rectangular co-ordinates in the horizontal plane and let t denote the time. We shall assume the motion to be governed by linearized shallow-water wave theory, in which the equations of motion are

$$\left.\begin{aligned} \frac{\partial u}{\partial t}-fv &= -g\frac{\partial \zeta}{\partial x}, \\ \frac{\partial v}{\partial t}+fu &= -g\frac{\partial \zeta}{\partial y} \end{aligned}\right\} \tag{2.1}$$

and the equation of continuity is

$$\frac{\partial \zeta}{\partial t} = -h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{2.2}$$

Here (u,v) denote the components of velocity in the (x,y) directions; ζ is the elevation of the free surface above the equilibrium level; f is the Coriolis parameter, g the acceleration of gravity and h the equilibrium depth. At a discontinuity in depth we shall assume that the surface elevation and the normal component of the flux are continuous; thus

$$\Big[\zeta\Big]_2^1 = 0, \quad \Big[h\mathbf{u}\,.\,\mathbf{n}\Big]_2^1 = 0, \tag{2.3}$$

where n denotes the unit normal to the discontinuity.

We seek solutions periodic in the time t. Then writing $u, v, w \propto e^{-\sigma t}$ where σ denotes the frequency, we have

$$\left.\begin{aligned} i\sigma u+fv &= g\frac{\partial \zeta}{\partial x}, \\ -fu+i\sigma v &= g\frac{\partial \zeta}{\partial y}, \end{aligned}\right\} \tag{2.4}$$

$$i\sigma\zeta = h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{2.5}$$

Solving for u and v from (2.4) we have (provided $\sigma^2 \neq f^2$)

$$\left.\begin{aligned} u &= \frac{1}{\sigma^2-f^2}\left(-i\sigma\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)g\zeta, \\ v &= \frac{1}{\sigma^2-f^2}\left(-f\frac{\partial}{\partial x}-i\sigma\frac{\partial}{\partial y}\right)g\zeta \end{aligned}\right\} \tag{2.6}$$

and on substituting in (5) we obtain the differential equation for ζ

$$\left(\nabla^2+\frac{\sigma^2-f^2}{gh}\right)\zeta = 0, \tag{2.7}$$

where ∇^2 denotes the two-dimensional Laplacian $(\partial^2/\partial x^2+\partial^2/\partial y^2)$. The boundary conditions (2.3) take the form

$$\Big[\zeta\Big]_1^2 = 0, \quad \left[h\left(-i\sigma\frac{\partial\zeta}{\partial x}+f\frac{\partial\zeta}{\partial y}\right)\right]_2^1 = 0 \tag{2.8}$$

if the x-axis is taken normal to the boundary.

We may have wave-like solutions

$$\zeta \propto \exp\{i(l'x+my-\sigma t)\} \tag{2.9}$$

to equation (2.7) provided that the wavenumbers l' and m satisfy

$$l'^2+m^2 = \frac{\sigma^2-f^2}{gh}. \tag{2.10}$$

It may be pointed out that the general tendency of the Coriolis term on the right of equation (2.10) is towards exponential rather than sinusoidal behaviour. Indeed if $f^2 > \sigma^2$ then l' and m cannot both be real, and the motion must vary exponentially in at least one horizontal direction.

Consider then instead of (2.9) the solution

$$\zeta \propto \exp\{-lx+i(my-\sigma t)\}, \tag{2.11}$$

which varies sinusoidally in the y-direction but exponentially (l being real) in the x-direction. In place of (2.10) we have

$$-l^2+m^2 = \frac{\sigma^2-f^2}{gh}. \tag{2.12}$$

The component of velocity in the x-direction is given by

$$u = \frac{1}{\sigma^2-f^2}(\sigma l+fm)ig\zeta. \tag{2.13}$$

The Kelvin-wave solution, in which

$$m = \pm\sigma/(gh)^{\frac{1}{2}}, \quad l = \mp f/(gh)^{\frac{1}{2}}, \tag{2.14}$$

is well known. This choice of m and l ensures that u vanishes identically; the component of the pressure gradient in the x-direction is exactly balanced by the Coriolis force.

To fix the ideas let us suppose m and f to be both positive. Then if σ is to be positive (so that the waves progress in the positive y-direction) l must be negative, that is, the waves decay exponentially to the left of the direction of propagation. This is a consequence of the conditions (2.13) that the transverse component of the velocity vanish. If u is not required to vanish it is possible to have waves which decay exponentially to the right of direction of propagation ($l > 0$). In the following section we shall consider the possibility of joining two such solutions, which progress along a discontinuity in depth and which decay exponentially away from the discontinuity on either side.

3. Waves trapped along a discontinuity

Let axes be chosen as in figure 1, with the x-axis normal to the discontinuity and the y-axis along the discontinuity itself. The mean depths to the left and right of the discontinuity are denoted by h_1 and h_2. We seek a solution in the form

$$\zeta \propto \begin{cases} \exp\{l_1 x + i(my - \sigma t)\} & (x < 0), \\ \exp\{l_2 x + i(my - \sigma t)\} & (x > 0), \end{cases} \tag{3.1}$$

where
$$l_1 > 0, \quad l_2 < 0. \tag{3.2}$$

From equation (2.12), we have in general† to satisfy

$$\left.\begin{aligned} -l_1^2 + m^2 &= \frac{\sigma^2 - f^2}{gh_1}, \\ -l_2^2 + m^2 &= \frac{\sigma^2 - f^2}{gh_2}. \end{aligned}\right\} \tag{3.3}$$

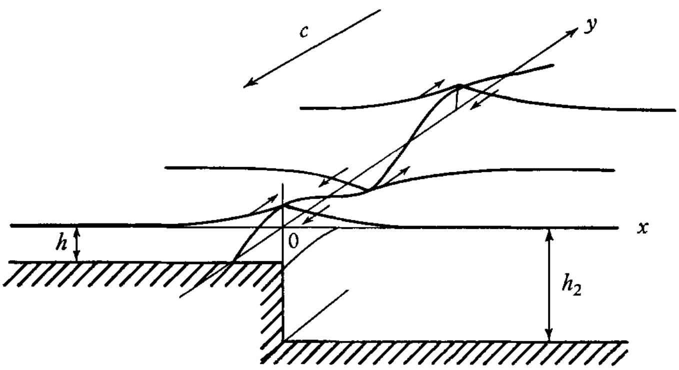

FIGURE 1. Sketch of physical situation.

The expressions (3.1) already satisfy the condition that ζ be continuous at $x = 0$, and from (2.13) the continuity of flux normal to the boundary requires that

$$h_1(\sigma l_1 - mf) = h_2(\sigma l_2 - mf). \tag{3.4}$$

For convenience we choose units in which

$$m = 1, \quad f = 1 \tag{3.5}$$

and write
$$h_2/h_1 = \gamma > 1; \quad f^2/m^2 g h_2 = \epsilon > 0. \tag{3.6}$$

Then equations (3.3) and (3.4) become

$$\left.\begin{aligned} l_1^2 - 1 &= \epsilon\gamma(1 - \sigma^2), \\ l_2^2 - 1 &= \epsilon(1 - \sigma^2), \\ (\sigma l_1 - 1) &= \gamma(\sigma l_2 - 1). \end{aligned}\right\} \tag{3.7}$$

We propose now to examine the range of the parameters y and ϵ for which solutions l_1, l_2, σ to (3.7) exist satisfying the conditions (3.2).

† That is to say if $\sigma^2 \neq f^2$. If $\sigma^2 = f^2$ it can easily be shown that there are no solutions to equations (2.1), (2.2) and (2.3) of the type required.

As a first step it is convenient to derive an identical relation connecting l_1, l_2 and σ, independently of γ and ϵ. Thus on eliminating γ and ϵ from equations (3.7) we have

$$(l_1^2-1)(\sigma l_2-1)-(l_2^2-1)(\sigma l_1-1) = 0. \tag{3.8}$$

This can be written

$$(l_1-l_2)(\sigma l_1 l_2-l_1-l_2+\sigma) = 0. \tag{3.9}$$

Since (l_1-l_2) cannot vanish, by equations (3.2), we may divide by this factor, and on writing

$$\sigma = -1/\tau, \tag{3.10}$$

(so that $|\tau|$ denotes the period of the oscillation in pendulum-days) we obtain the symmetrical relation

$$l_1 l_2+l_1\tau+l_2\tau+1 = 0. \tag{3.11}$$

In the space of the co-ordinates (l_1, l_2, τ) this represents a quadric surface which can be reduced to normal form by the substitution

$$\left.\begin{aligned} l_1 &= \xi+\eta-\tau, \\ l_2 &= -\xi+\eta-\tau, \end{aligned}\right\} \tag{3.12}$$

or inversely

$$\left.\begin{aligned} \xi &= \tfrac{1}{2}(l_1-l_2), \\ \eta &= \tfrac{1}{2}(l_1+l_2)+\tau. \end{aligned}\right\} \tag{3.13}$$

Equation (3.11) then becomes

$$\xi^2-\eta^2+\tau^2 = 1 \tag{3.14}$$

from which it is clear that the surface is a hyperboloid of one sheet.

Now the third of equations (3.7) can be written

$$(l_1-\tau) = \gamma(l_2-\tau), \tag{3.15}$$

which in the new co-ordinates becomes

$$\frac{\eta}{\xi} = \frac{\gamma+1}{\gamma-1} > 1. \tag{3.16}$$

This represents a plane through the τ-axis. From (3.14) it follows immediately that

$$\tau^2-1 = \eta^2-\xi^2 > 1. \tag{3.17}$$

Therefore $|\tau|$ always exceeds unity. Hence the wave period is always greater than a pendulum-day.

By use of the new co-ordinates one can also obtain two simple relations involving only ξ, τ and one or other of the parameters γ and ϵ. Essentially we do this by considering the projection on the (ξ, τ)-plane of the intersections of the plane (3.16) with the other surfaces of (3.7) and (3.14). Thus from (3.16) and (3.14) we have on eliminating η

$$\xi^2\left[1-\left(\frac{\gamma+1}{\gamma-1}\right)^2\right]+\tau^2 = 1, \tag{3.18}$$

whence

$$\xi = \pm\frac{\gamma-1}{2\gamma^{\frac{1}{2}}}(\tau^2-1)^{\frac{1}{2}}. \tag{3.19}$$

Similarly from the second of equations (3.7) we have

$$(-\xi+\eta-\tau)^2 = 1+\epsilon(1-\tau^{-2}) \tag{3.20}$$

and again eliminating η by (3.16) we obtain

$$\xi = \tfrac{1}{2}(\gamma-1)\{\tau \pm [1+\epsilon(1-\tau^{-2})]^{\frac{1}{2}}\}. \tag{3.21}$$

Writing

$$\frac{2}{\gamma-1}\xi = \xi' \tag{3.22}$$

we then have the pair of relations

$$\xi' = \pm\gamma^{-\frac{1}{2}}(\tau^2-1)^{\frac{1}{2}} = F(\gamma,\tau), \tag{3.23}$$

$$\xi' = \tau \pm [1+\epsilon(1-\tau^{-2})]^{\frac{1}{2}} = G(\epsilon,\tau). \tag{3.24}$$

These give us two families of curves whose intersections will correspond to possible wave periods $|\tau|$.

However, we must first take account of the restrictions imposed by the two inequalities (3.2). These can be simply expressed in terms of the new co-ordinates by (3.12); on using (3.15) we find that $l_1 > 0$ and $l_2 < 0$ imply respectively

$$\gamma\xi' > \tau \quad \text{and} \quad \xi' < \tau. \tag{3.25}$$

Since $\gamma > 1$, the only possibility is that ξ' and τ are both positive. Hence in (2.33) we must take the positive sign and again, since

$$\gamma^{-\frac{1}{2}}(\tau^2-1)^{\frac{1}{2}} < (\tau^2-1)^{\frac{1}{2}} < \tau,$$

the only possible intersections of (3.24) with (3.23) correspond to the negative sign of the radical in (3.24).

In figure 2 we have plotted the two families of curves $\xi' = F(\gamma,\tau)$ and $\xi' = G(\epsilon,\tau)$ when ξ' and τ are both positive. The limiting curve $\xi' = F(1,\tau)$ is also shown. Since, for large values of $|\tau|$,

$$F(\gamma,\tau) \sim \gamma^{-\frac{1}{2}}\tau, \quad G(\epsilon,\tau) \sim \tau, \tag{3.26}$$

it is clear that the curve $\xi' = F$ must in general intersect the curve $\xi' = G$ in at least one point apart from $\xi' = 0$, $\tau = 1$. On the other hand since the second derivatives of F and G are of opposite sign there can be only one such intersection. To see that this point of intersection satisfies (3.25) we note that (3.25) together with (3.23) implies that

$$\xi' < \tau-(1/\tau). \tag{3.27}$$

Remembering that ξ' and τ are both positive we see that the intersections have to lie below the dotted curve shown in figure 2. This the intersections always do, since they must always satisfy (3.24) with $\epsilon > 0$.

Three limiting cases are of interest. First, when $\gamma = 1$ the curve $\xi' = F(1,\tau)$ intersects none of the curves $\xi' = G(\epsilon,\tau)$ (save at infinity). Hence there are no trapped waves, as we should expect since $\gamma = 1$ corresponds to $h_1 = h_2$ and hence uniform depth h.

Second, when $\gamma = \infty$, the function $F(\gamma,\tau)$ vanishes and so the solutions are given by

$$G(\epsilon,\tau) = 0. \tag{3.28}$$

Hence

$$(\tau^2-1)(1-\epsilon/\tau^2) = 0. \tag{3.29}$$

Disregarding the case $\tau^2 = 1$, that is $\sigma^2 = f$, we have

$$\tau = \epsilon = (gh)^{-\frac{1}{2}} \tag{3.30}$$

(m and f being equal to unity). These correspond to Kelvin waves, as we expect since $h_1/h_2 \to 0$. From figure 2 it is clear that the wave period τ for general values of γ always exceeds, or at least equals, the Kelvin wave period τ_0 even when $\epsilon > 1$. On the other hand when $\epsilon < 1$, so that the Kelvin wave period is less than a pendulum-day, the wave period τ has the lower bound 1 as shown previously.

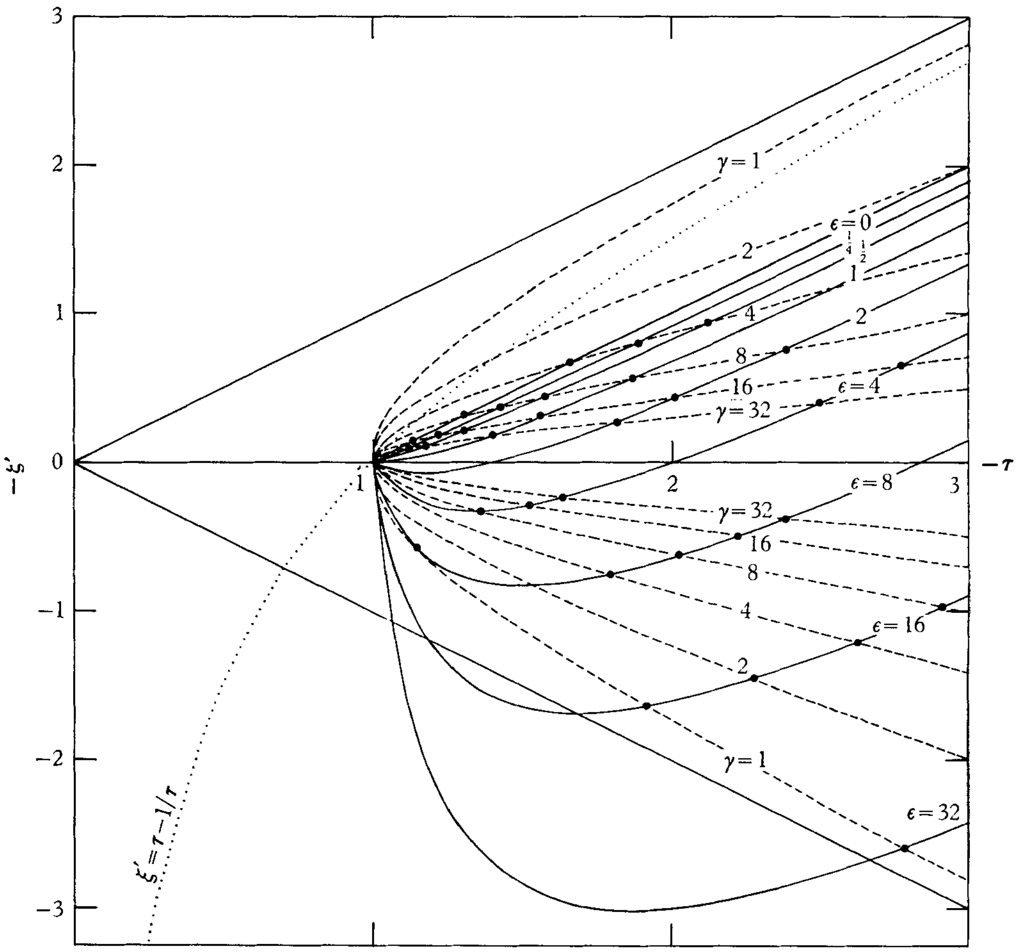

FIGURE 2. The functions $\xi' = F(\gamma, \tau)$ (broken curves) and $\xi' = G(\gamma, \tau)$ (solid curves), whose intersection gives the wave period $|\tau|$.

The third limiting case is when $\epsilon \to 0$. These correspond to non-divergent planetary waves, the special case already discovered by Rhines (see Bretherton *et al.* 1966). We have then from (3.24)

$$G(\epsilon, \tau) \to G(0, \tau) = \tau - 1 \tag{3.31}$$

and so on equating (3.23) and (3.24)

$$(\tau^2 - 1) = \gamma(\tau - 1)^2. \tag{3.32}$$

Again ignoring the case $\tau = 1$ we have

$$(\tau + 1) = \gamma(\tau - 1), \tag{3.33}$$

and so

$$\tau = \frac{\gamma+1}{\gamma-1}. \tag{3.34}$$

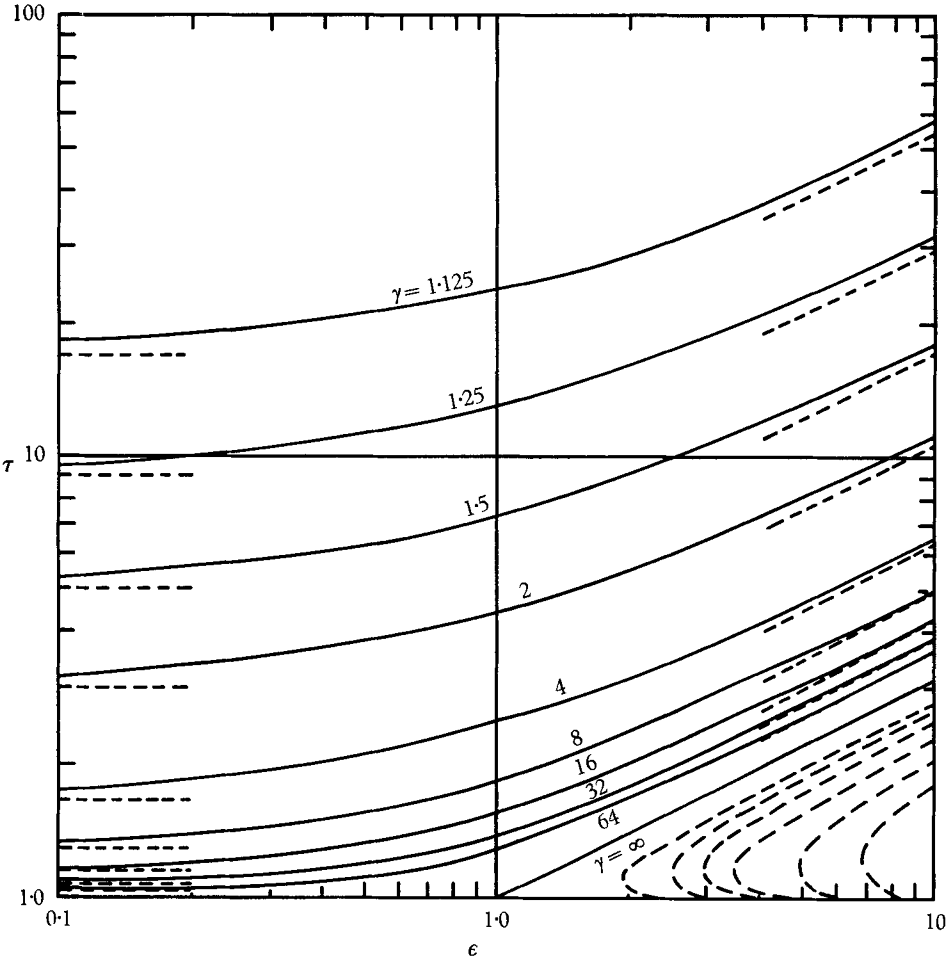

FIGURE 3. The wave period τ in pendulum-days, as a function of the parameter $\alpha = f^2/m^2gh_2$, where m = wave-number, f = Coriolis parameter, g = gravity and h_2 = mean depth on deeper side. The curves are shown for constant values of γ, the ratio of the depths.

The corresponding values of l_1 and l_2 may be found from equations (3.7). They are simply

$$l_1 = 1, \qquad l_2 = -1. \tag{3.35}$$

In other words the wave amplitude in this case decays exponentially at the same rate on either side of the discontinuity. The modulus of decay is equal to unity, or, in real units, to the wave-number m.

In general, since τ is always positive, σ is negative, and it follows that the phase velocity σ/m is negative, or the waves are necessarily propagated with the discontinuity on the right of the direction of propagation, regarded from deep water. (When f is negative, the discontinuity is on the left.)

Hence all the predictions made in §2 concerning the period and the direction of propagation have been verified.

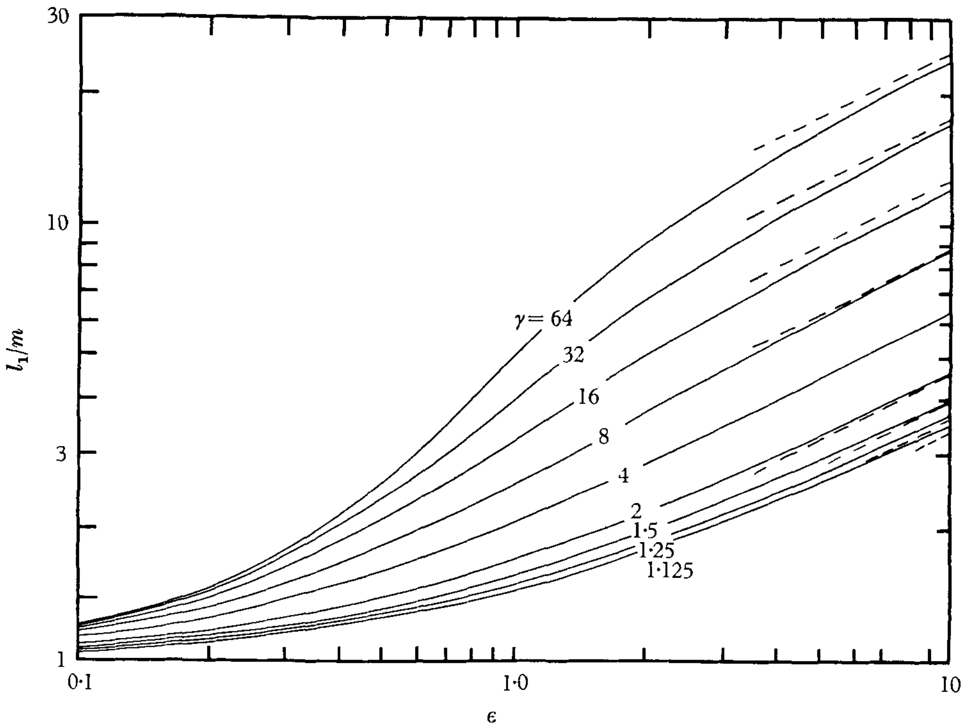

FIGURE 4. The proportional rate of decay l_1 on the shallower side of the discontinuity, shown as a function of ϵ for given values of γ, $= h_2/h_1$(see legend to figure 2).

Some explicit analytical expressions for the wave period $|\tau|$ as a function of γ and ϵ, are derived in the appendix. In figure 3, $|\tau|$ has been plotted as a function of ϵ for fixed values of γ. The corresponding values of the exponential decrements l_1 and $-l_2$ are shown in figures 4 and 5. From (3.7) it is clear that we must always have

$$l_1 > -l_2 > 1. \tag{3.36}$$

Figure 3 also shows that $|\tau|$ is an increasing function of ϵ. Hence $-\sigma$ is an increasing function of m and so the group-velocity is in the same direction as the phase velocity. As $\epsilon \to 1$ and $\gamma \to \infty$, so the group-velocity tends to infinity.

The tangential component of velocity near the discontinuity is found from equations (2.6) to be given by

$$v = \frac{\sigma m - fl}{\sigma^2 - f^2} g\zeta, \tag{3.37}$$

where $l = l_1$ or l_2 according as $x \lessgtr 0$. In non-dimensional units we therefore have

$$v_1 = -\frac{\sigma - l_1}{1-\sigma^2} g\zeta, \quad v_2 = -\frac{\sigma - l_2}{1-\sigma^2} g\zeta. \tag{3.38}$$

Now from (3.36), since $-1 < \sigma < 0$, we have

$$-(\sigma - l_1) > 1 > 0, \qquad -(\sigma - l_2) < 0. \tag{3.39}$$

Therefore the velocity at the wave crests is forwards (in the direction of wave propagation) on the deeper side, and backwards on the shallower side. The discontinuity in the velocity is given by

$$v_2 - v_1 = \frac{l_2 - l_1}{1 - \sigma^2} g\zeta. \tag{3.40}$$

So from (3.36)

$$|v_2 - v_1| > 2|g\zeta|. \tag{3.41}$$

We see then that our model necessarily implies a vortex sheet, of alternating sign, in the plane $x = 0$.

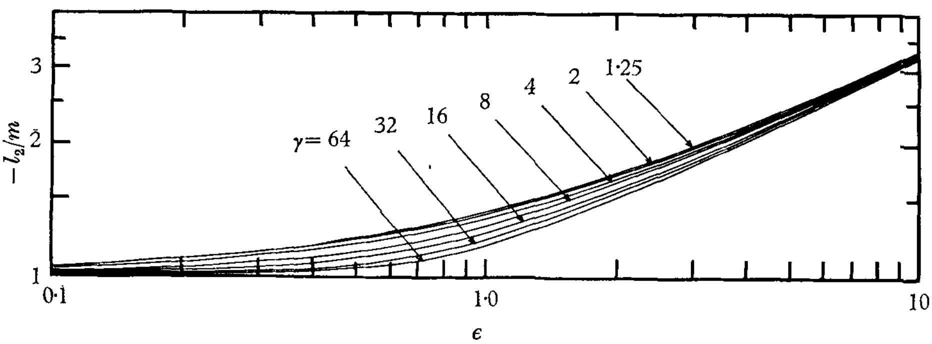

FIGURE 5. The proportional rate of decay $-l_2$ on the deeper side of the discontinuity, shown as a function of ϵ for given values of γ (see legend to figure 2).

4. The reflexion of waves at a discontinuity

It may be of interest to examine briefly the conditions under which waves in one medium may be totally reflected at the discontinuity. From equation (2.7) it follows that if waves are to be propagated without attenuation on either one side or other of the discontinuity, then we must have

$$\sigma^2 \geqslant f^2, \qquad \tau^2 \leqslant f^{-2}, \tag{4.1}$$

in other words the period must be less than a pendulum-day. There is thus no possibility of such motions overlapping with the trapped wave motions, all of whose periods must exceed a pendulum-day.

A wave incident from the side $x < 0$ (water of depth h_1) and being partially reflected and partially transmitted may be represented by the expressions

$$\zeta = \begin{cases} (\cos l_1' x + A \sin l_1' x) \exp\{i(my - \sigma t)\} & (x < 0), \\ \exp\{l_2 x + i(my - \sigma t)\} & (x > 0), \end{cases} \tag{4.2}$$

where l_1' and l_2 are given (in non-dimensional units) by

$$\left.\begin{aligned} -l_1'^2 &= \epsilon\gamma(1-\sigma^2) + 1, \\ l_2^2 &= \epsilon(1-\sigma^2) + 1 \end{aligned}\right\} \tag{4.3}$$

and A is a constant to be determined. l_1' must be real, l_2 may be real or imaginary, but with $\mathscr{R}(l_2) < 0$. The assumed forms (4.2) satisfy the continuity of ζ at $x = 0$, and the continuity of the normal flux is satisfied provided that

$$h_1(-\sigma l_1' A + fm) = h_2(-\sigma l_2 + fm). \tag{4.4}$$

Thus in non-dimensional units

$$A = \frac{-\gamma l_2 + (\gamma - 1)/\sigma}{l_1'}. \tag{4.5}$$

In the present section it has not yet been specified whether h_1 is greater or less than h_2, so that γ may be less or greater than 1. However, on subtracting the two equations (4.3) we have

$$-l_1'^2 - l_2^2 = \epsilon(\gamma - 1)(1 - \sigma^2). \tag{4.6}$$

But we have seen that $\sigma^2 > 1$. Hence if l_1' is to be real we must have $\gamma > 1$. In other words, total reflexion can take place only if the waves are incident from the shallower side.

The critical angle of incidence occurs when $l_2 = 0$. From (4.3) this gives

$$\epsilon(1 - \sigma^2) + 1 = 0 \tag{4.7}$$

and so

$$\sigma^2 = 1 + \epsilon^{-1}. \tag{4.8}$$

Again from (4.3) we have in that case

$$l_1'^2 = \gamma - 1. \tag{4.9}$$

Thus the tangent of the critical angle of incidence is given by

$$\frac{m}{l_1'} = \frac{1}{(\gamma - 1)^{\frac{1}{2}}}. \tag{4.10}$$

This is remarkable, in so far as the angle depends only on the ratio of the depths and is independent of the Coriolis parameter f.

By a converse of the conclusion derived previously from (4.6), we see that if a wave is incident on the ridge from the deeper side it must be at least partially transmitted. The reflexion coefficient can never vanish; for if the wave is to be transmitted l_2 must be purely imaginary while σ is real. Hence the numerator of equation (4.5) cannot vanish, and indeed A must be complex. The amplitude of the reflected wave is equal to $\frac{1}{2}|1 + iA|$, which can then not be zero.

It may be asked whether a wave propagated from infinity and incident on the ridge from either side (but not exactly along the ridge) would be capable of setting up any trapped waves; in other words whether any of the incident energy could be captured by the discontinuity.

Clearly if the wave is periodic in time such capture is impossible; for the period of the incident waves must be less than a pendulum-day while the period of the trapped waves is greater than a pendulum-day.

A limited periodic disturbance with period greater than a pendulum-day could not normally be propagated from infinity without attenuation (as shown in §2). However, if the initial disturbance takes place in the neighbourhood of a discontinuity, the latter provides a suitable channel along which wave energy may be guided for an indefinite distance, without any attenuation save that due to viscous and other effects.

5. Conclusions

We have shown that it is theoretically possible for waves to be trapped along a discontinuity in depth, the wave amplitude decreasing exponentially to either side. The waves always travel in the direction of Kelvin waves in the deeper fluid, that is to say having the discontinuity to their right in the northern hemisphere to their left in the southern hemisphere. Their period always exceeds the period of the corresponding Kelvin waves, to which they may tend whenever the Kelvin wave period exceeds the inertial period; that is to say whenever

$$\frac{f^2}{m^2 g h_2} > 1. \tag{5.1}$$

If the above inequality is not satisfied then the lower limit of the wave period is equal to the inertial period.

An appropriate name for the type of motion described above would be a 'double Kelvin wave', or more picturesquely a 'seascarp' wave.

It has been shown also that propagating waves, because their period is less than the inertial period, cannot be captured by a discontinuity in depth, although they may be totally reflected if they approach from the shallower side. On the other hand periodic disturbances whose period is greater than the inertial period, provided they occur in the neighbourhood of the discontinuity (that is to say within a distance of order $[gh/(f^2-\sigma^2)]^{\frac{1}{2}}$) are capable of generating trapped waves, and these may be channelled for great distances along the discontinuity.

It is interesting to speculate what will happen if the discontinuity gradually disappears, the depth becoming uniform. Since wave energy cannot be propagated across a region of uniform depth, presumably the wave energy will accumulate in the transition zone until an instability occurs.

To investigate this idea more closely, consider waves of a particular period τ. These will correspond to points along a particular horizontal line τ = constant in figure 3. As the contrast in depth is diminished so $\gamma = h_2/h_1$ decreases. Since the curves in figure 3 represent monotonically varying functions, ϵ must also decrease monotonically, that is to say the representative point moves to the left along the line τ = constant. A limit is reached when $\epsilon \to 0$, corresponding to non-divergent waves. Then, as we have seen,

$$\tau = \frac{\gamma+1}{\gamma-1}, \quad \text{so} \quad \gamma = \frac{\tau+1}{\tau-1}. \tag{5.2}$$

Clearly, the greater the period, the smaller is the corresponding value of γ and so the further the waves can be propagated along the discontinuity.

It is easy to show that for non-divergent waves ($\epsilon \to 0$) the group velocity tends to zero. The flux of energy being independent of position, it follows that the energy density even in a wave group of finite length must tend to infinity (within the limits of the linear theory). Hence, we may expect that, if a continuous spectrum of waves is propagated along a diminishing discontinuity, at any given point the corresponding periodicity given by (5.2) will be prominent. The discontinuity thus acts as a kind of chromatograph.

Because the limiting waves are non-divergent, this effect would appear in a record of horizontal currents, rather than pressure or surface elevation.

At the same time it may be noted that as $\epsilon \to 0$ so the wave-number m becomes large (again within the limits of the theory); and since the exponential decrements l_1 and $-l_2$ are always greater than m it follows that the waves are confined to a narrower and narrower region on either side of the discontinuity.

One aspect of our model theory may be thought questionable, namely, the presence of a singularity in the tangential flow along the discontinuity. This results directly from our assumed boundary conditions, that the surface elevation and the normal component of the total flow are to be continuous at the boundary. These conditions have been justified for a special example by Bartholomeusz (1958) in a system without rotation. In the presence of rotation the assumptions have not yet been justified so rigorously. However, since the strength of the vortex sheet will vary sinusoidally in time, there seems no reason to suppose that, at small amplitudes, much vorticity will be propagated away from the discontinuity. Nevertheless, the high rate of shear will result in an additional loss of energy by viscosity, apart from that already existing at the solid boundaries.

It may not be necessary to assume that the depth is discontinuous, and hence it would be interesting to investigate the possibility of trapped wave motions when the depth undergoes a more or less smooth transition from one uniform value (when $x < -\delta$, say), to a second uniform value (when $x > \delta$). Even with a continuous transition, similarly trapped modes are to be expected, which will become double Kelvin waves as the width 2δ of the transition zone tends to zero.

Some attention should now be given to the problem of the forced excitation of trapped waves by normal atmospheric pressures or horizontal wind stresses. One may expect that the former will be more effective over deep water, and the latter more effective over the shallower water of a continental shelf. From the fact that the tangential velocity is of opposite sign on the two sides of the discontinuity one may expect the most favourable conditions for wave generation to arise when the wind is parallel to the discontinuity and in opposite directions on the two sides of the discontinuity. However, even if the tangential wind stress is uniform in strength and direction it must generate a larger *mean* current in the shallower water. The difference in the mean currents on the two sides of the discontinuity will then tend to set up wave motions propagated along the discontinuity as before.

One might at first sight imagine that baroclinic wave motions of a similar type could occur in stratified fluids, along horizontal paths where the density stratification of the fluid undergoes a change in vertical structure. However, such a system would not be in dynamical equilibrium without the presence of steady transverse currents to balance the horizontal gradients of pressure. The presence of currents would in turn alter the dynamics of the small perturbations and probably would result in unstable waves similar to those which occur at atmospheric fronts. This phenomenon, though interesting in itself is outside the scope of the present paper.

Finally, we should like to make a reference to two papers by the Russian mathematician Voit (1961, 1963). In the first of these (Voit 1961) the author treated the problem of tidal waves propagated from the mouth of a channel into a semi-infinite plane, in which there was a depth discontinuity normal to the direction of the channel. This situation might have been expected to produce waves more or less trapped at the discontinuity, if the wave period had been greater than the inertial period, and if the discontinuity had been sufficiently close to the mouth of the channel. However, the only case considered in detail was that of the semi-diurnal tide, for which the period necessarily does not exceed the inertial wave period; and thus no waves trapped at the discontinuity were found.

In the second paper (Voit 1963) the author considered the transient disturbance due to an initial elevation of the free surface, in an infinite basin in which there are two parallel discontinuities in depth. Again no trapped waves were found, due to the fact that the form of the initial disturbance was uniform in a direction parallel to the discontinuities. Had the initial disturbance been taken at right angles to the discontinuities, or inclined at some general angle, the phenomenon of wave trapping must have appeared.

Appendix: Explicit expressions for τ

In order to derive explicit expressions for the wave period $|\tau|$, it is convenient to introduce instead of γ and ϵ two parameters α and ω which remain unaltered when h_1 and h_2 are interchanged, namely

$$\alpha = \frac{1}{2}\left[\left(\frac{h_2}{h_1}\right)^{\frac{1}{2}} + \left(\frac{h_1}{h_2}\right)^{\frac{1}{2}}\right] = \frac{\gamma^{\frac{1}{2}}+\gamma^{-\frac{1}{2}}}{2} \tag{A 1}$$

and

$$\omega = \frac{f^2}{2m^2(h_1 h_2)^{\frac{1}{2}}} = \frac{\gamma^{\frac{1}{2}}\epsilon}{2}. \tag{A 2}$$

Inversely, we have

$$\gamma^{\frac{1}{2}} = \alpha + (\alpha^2-1)^{\frac{1}{2}}, \quad \epsilon = \frac{2\omega}{\alpha+(\alpha^2-1)^{\frac{1}{2}}}. \tag{A 3}$$

Now let equations (3.7) be written in the form†

$$\left.\begin{aligned} \gamma^{-1}l_1^2 &= 2\omega\gamma^{-\frac{1}{2}}(1-\sigma^2)+\gamma^{-1}, \\ \gamma l_1^2 &= 2\omega\gamma^{\frac{1}{2}}(1-\sigma^2)+\gamma \end{aligned}\right\} \tag{A 4}$$

and

$$\gamma^{-\frac{1}{2}}l_1 + \gamma^{\frac{1}{2}}l_2 = 2(1-\alpha^2)/\sigma. \tag{A 5}$$

To eliminate l_1 and l_2 we may first square both sides of (A 5) and substitute for $\gamma^{-1}l_1^2$, and γl_2^2 from (A 4), giving

$$2l_1 l_2 = 4[\alpha(\alpha-\omega\sigma^2)+1](1-\sigma^2)/\sigma^2+2. \tag{A 6}$$

† Equation (A 7) below can also be derived by equating the functions F and G of § 3; equations (A 4) to (A 6) represent an alternative.

On squaring again and substituting again from (A 5) we find, after division by $(1-\alpha^2)(1-\sigma^2)$, the relation

$$(\alpha-\omega\sigma^2)^2(1-\sigma^2)-1 = 0, \tag{A 7}$$

that is

$$\frac{1}{(\alpha-\omega\sigma^2)^2} = 1-\sigma^2. \tag{A 8}$$

On writing

$$\frac{1}{\alpha-\omega\sigma^2} = \lambda, \qquad \sigma^2 = \frac{1-\alpha\lambda}{\omega\lambda}, \tag{A 9}$$

equation (A 7) reduces to the simple form

$$\omega\lambda^3-(\omega-\alpha)\lambda-1 = 0. \tag{A 10}$$

In addition, (A 8) shows that

$$\lambda^2 = 1-\sigma^2, \quad \text{so} \quad \tau^2 = 1/(1-\lambda^2). \tag{A 11}$$

The solution to the cubic equation (A 10) depends, as is well known, on the relative signs of the coefficients ω and $(\omega-\alpha)$ and also on the magnitude of $(\omega-\alpha)^3/\omega$. In our application, ω is always positive, and so we have the following cases.

(i) If $(\omega-\alpha) \geqslant 0$, then the solution is given by

$$\left.\begin{aligned} &\lambda = r\cos\theta, \\ \text{where} \quad &r = \left[\frac{4(\omega-\alpha)}{3\omega}\right]^{\frac{1}{2}} \\ \text{and} \quad &3\theta = \cos^{-1}\left(\frac{4}{\omega r^3}\right) \quad \text{or} \quad \cosh^{-1}\left(\frac{4}{\omega r^3}\right) \end{aligned}\right\} \tag{A 12}$$

according as $\omega r^3/4 \lesseqgtr 1$. In the critical case when

$$\omega r^3/4 = 1, \quad \text{that is} \quad 4(\omega-\alpha)^3 = 27\omega,$$

then clearly

$$\lambda = (4/\omega)^{\frac{1}{3}}, \quad -(1/2\omega)^{\frac{1}{3}}, \quad -(1/2\omega)^{\frac{1}{3}},$$

two of the roots being coincident.

(ii) If $(\omega-\alpha) < 0$ then the solution is given by

$$\lambda = r\sinh\theta,$$

where

$$r = \left[\frac{4(\alpha-\omega)}{3\omega}\right]^{\frac{1}{2}}, \quad 3\theta = \sinh^{-1}\left(\frac{4}{\omega r^3}\right). \tag{A 13}$$

These are the analytical formulae promised earlier. A simple way to visualize the roots is as follows. Let

$$\omega^{\frac{1}{3}}\lambda = \mu \tag{A 14}$$

so that (A 10) reduces to the even simpler form

$$\mu^3-3\beta\mu-1 = 0 \tag{A 15}$$

with

$$\beta = (\omega-\alpha)/3\omega^{\frac{2}{3}}. \tag{A 16}$$

The behaviour of μ as a function of β can be seen by solving (A 15) for β and plotting the function

$$\beta = (\mu^3-1)/3\mu, \tag{A 17}$$

as in figure 6. The asymptotes $\beta = \mu^3/3$ and $\beta = -1/(3\mu)$ are also shown. Clearly β has a minimum value where $\mu = -2^{-\frac{1}{3}}$ and $\beta = 2^{-\frac{2}{3}}$. If β exceeds this value μ has three roots; if β is less than this value μ has only one root. It can be shown, however, that in the former case either $l_1 < 0$ or $l_2 > 0$. Hence for trapped waves we are limited to the upper branch of the curve in figure 6.

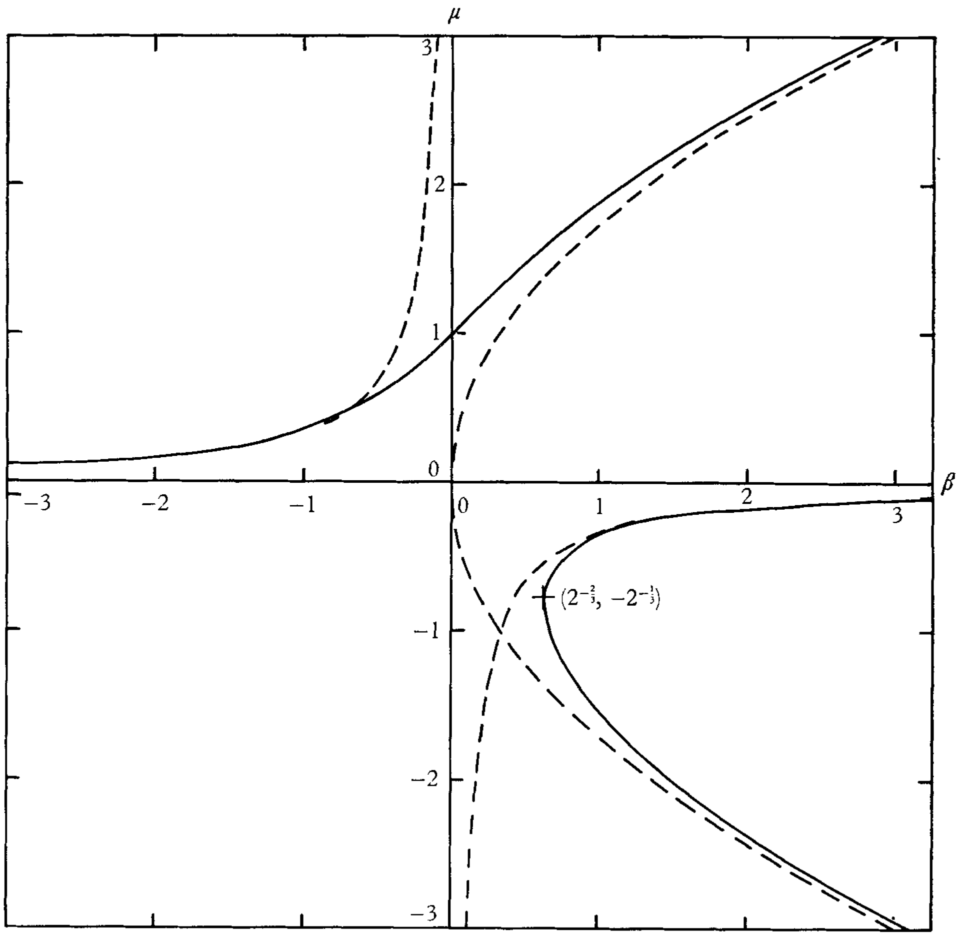

Figure 6. Curves giving the roots of the cubic equation (A 15) as a function of β.

Suppose now that we wish to plot the wave period τ as a function of the parameter ϵ (proportional to m^{-2}) for given values of γ, $= h_2/h_1$. Let us begin by plotting λ, $= (1-1/\tau^2)$ as a function of $\omega(=\frac{1}{2}\gamma^{\frac{1}{2}}\epsilon)$ for given values of α, $= \frac{1}{2}(\gamma^{\frac{1}{2}}+\gamma^{-\frac{1}{2}})$. Equation (A 16) shows that the parameter

$$\beta = \tfrac{1}{3}(\omega^{\frac{2}{3}} - \alpha\omega^{-\frac{1}{3}}) \tag{A 18}$$

is a monotonically increasing function of ω over the range $0 < \omega < \infty$. Likewise equation (A 14) shows that

$$\lambda = \omega^{-\frac{1}{3}}\mu \tag{A 19}$$

is a monotonically decreasing function of ω. Hence the curves of $\lambda(\omega)$ for given values of γ have a form similar to that of figure 6.

It is now simple to make the transition to the curves of $\tau(\epsilon)$ for fixed values of γ. These are shown in figures 3–5 (see §3).

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REFERENCES

Bartholomeusz, E. F. 1958 The reflexion of long waves at a step. *Proc. Cambridge Phil. Soc.* **54**, 104–8.

Bretherton, F. P., Carrier, G. F. & Longuet-Higgins, M. S. 1966 Report on the I.U.T.A.M. symposium on rotating fluid systems, p. 404. *J. Fluid Mech.* **26**, 393–410.

Eckart, C. 1951 Surface waves on water of variable depth. *Wave Rept.* no. 100, *U.S. Office of Naval Research* (Notes of lectures given at the Scripps Institution of Oceanography.)

Longuet-Higgins, M. S. 1967 On the trapping of wave energy round islands. *J. Fluid Mech.* **29**, 781–821.

Mysak, L. A. 1967 On the theory of continental shelf waves. *J. Mar. Res.* **25**, 205–227.

Robinson, A. R. 1964 Continental shelf waves and the response of the sea surface to weather systems. *J. Geophys. Res.* **69**, 367–8.

Snodgrass, F. E., Munk, W. H. & Miller, G. R. 1962 Long-period waves over California's continental borderland. Part 1. Background spectra. *J. Mar. Res.* **20**, 3–30.

Stokes, G. G. 1846 Report on recent researches in hydrodynamics. *Brit. Ass. Ref.* 1846; see also *Coll. Pap.* **1**, 167.

Thomson, W. 1879 On gravitational oscillations of rotating water. *Proc. Roy. Soc. Edinburgh*, **10**, 92.

Ursell, F. 1952 Edge waves on a sloping beach. *Proc. Roy. Soc.* A**214**, 79–97.

Voit, S. S. 1961 The propagation of tidal waves from a strait into a basin of variable depth. *Trans. Mar. Hydrophys. Inst., U.S.S.R. Acad. Sci.* **24**, 89–104. *Am. Geophys Un. Translation* **24**, 83–98.

Voit, S. S. 1963 Propagation of transient long waves in a rotating basin of variable depth. *Trans. Mar. Hydrophys. Inst., U.S.S.R. Acad. Sci.* **27**, 11–25. *Am. Geophys. Un. Translation* **27**, 7–20.